ISSN 2686-9667. Вестник российских университетов. Математика

Том 26, № 134 2021

DOI 10.20310/2686-9667-2021-26-134-143-150
UDC 517.988.63, 519.216.2

A counterexample to the stochastic version of the Brouwer fixed point theorem

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Контрпример к стохастической версии теоремы Брауэра о неподвижной точке

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Abstract. It is shown that the stochastic counterpart of the classical fixed point theorem for continuous maps in a finite dimensional Euclidean space (“Brouwer’s theorem”) is not, in general, true. This result implies, in particular, that a careful choice of invariant sets in the stochastic version of Brouwer’s theorem is necessary in the theory of stochastic nonlinear operators.

Keywords: local operators, convergence in probability, fixed points

Mathematics Subject Classification: 34A9, 34K50, 46N20

Acknowledgements: The work is partially supported by the grant #239070 of the Research Council of Norway.

For citation: Ponosov A. Kontrprimer k stokhasticheskoy versii teoremy Brauera o nepodvizhnoy tochke [A counterexample to the stochastic version of the Brouwer fixed point theorem]. *Vestnik rossiyskikh universitetov. Matematika – Russian Universities Reports. Mathematics*, 2021, vol. 26, no. 134, pp. 143–150. DOI 10.20310/2686-9667-2021-26-134-143-150.

Аннотация. Показано, что стохастический аналог классической теоремы о неподвижной точке для непрерывных отображений в конечномерном евклидовом пространстве («теорема Брауэра»), вообще говоря, неверен. Этот результат означает, в частности, что в теории стохастических нелинейных операторов необходим тщательный выбор инвариантных множеств в стохастической версии теоремы Брауэра.

Ключевые слова: локальные операторы, сходимость по вероятности, неподвижные точки

Благодарности: Работа выполнена при поддержке гранта #239070 Исследовательского Совета Норвегии.

Для цитирования: *Поносов А.* Контрпример к стохастической версии теоремы Брауэра о неподвижной точке // Вестник российских университетов. Математика. 2021. Т. 26. № 134. С. 143–150. DOI 10.20310/2686-9667-2021-26-134-143-150. (In Engl., Abstr. in Russian)

Introduction

The classical Brouwer fixed point theorem says that if $V \subset R^n$ is closed, convex, bounded and nonempty, then any continuous operator $T: V \to V$ has at least one fixed point. This result is an important contribution to e. g. nonlinear functional analysis and its applications, where it is used to justify the fixed point theory for compact operators (Schauder's fixed point theorem) and their generalizations (Sadovskii's fixed point theorem [1] etc.). Existence problems in the theory of stochastic equations can also be formulated using the fixed point framework [2], [3]. However, there are several reasons why this framework cannot be based on Brouwer's theorem or its natural generalizations: the relevant spaces are not Banach, and even not locally convex, and the relevant operators are far from being compact. At the same time, stochastic operators possess some other generic properties, which may be of significance for the stochastic analysis [3].

In this paper, we describe the class of operators that typically stem from stochastic equations and discuss the assumptions on invariant sets that can be used in a potential fixed-point theorem for these operators. The main result of the paper gives a nontrivial counterexample of a closed, convex, bounded and nonempty subset, for which the stochastic Brouwer fixed point theorem, formulated for the above class of nonlinear operators, is not valid.

1. Local operators

Let $\mathcal{S} = (\Omega, \mathcal{F}, P)$ be a complete probability space, which means that a probability measure P is defined on a σ-algebra $\mathcal{F}$ of subsets of a set Ω, and $\mathcal{F}$ contains all subsets of measure 0. Below, the abbreviation a.s. replaces the expression "almost surely", i. e. almost everywhere with respect to the measure P.

For a separable metric space X, the set $\mathcal{P}(X)$ consists of all equivalence classes $[x]$ of $\mathcal{F}$-measurable functions $x: \Omega \to X$, also called random points in X. Equipped with the topology of convergence in probability, $\mathcal{P}(X)$ becomes a complete topological vector space, which is not locally convex even if $X = R^n$.

Let $\Xi \subset \mathcal{P}(R^n)$. We say that two equivalence classes $[x], [y] \in \Xi$ coincide on a set $A \subset \Omega$, i. e. $[x]|_A = [y]|_A$, if $x(\omega) = y(\omega)$ for almost all $\omega \in A$ for some representatives $x \in [x]$ and $y \in [y]$. Evidently, this definition is independent of the choice of the representatives x and y.

D e f i n i t i o n 1.1. Let $\Xi \subset \mathcal{P}(X)$. An operator $h: \Xi \to \mathcal{P}(Y)$, where Y is another separable metric space, is called **local** if

$$[x]|_A = [y]|_A \text{ implies } h[x]|_A = h[y]|_A$$

for any $[x], [y] \in \Xi$ and $A \subset \Omega$.

Notice that any local operator h can be naturally (but not uniquely) extended from the set Ξ to the set of all representatives of the equivalence classes belonging to Ξ. Indeed, for $[x] \in \Xi$ we can put hx to be an arbitrary representative of the class $h[x]$. Clearly, such an operator is well-defined. For this extension, the property of locality reads as follows:

$$x(\omega) = y(\omega) \text{ for } \omega \in A \text{ a.s. implies } hx(\omega) = hy(\omega) \text{ for } \omega \in A \text{ a.s.}$$

Conversely, if h, defined as a local operator on the set of all representatives of the equivalence classes belonging to Ξ, is local, then it generates a unique local operator on the set Ξ because of the property $x_1, x_2 \in [x]$ implies $h(x_1) = h(x_2)$ a.s. Therefore, we will in many cases disregard the difference between the equivalence classes $[x]$ and their particular representatives x writing (somewhat unprecisely) $x \in \mathcal{P}(X)$ instead of $[x] \in \mathcal{P}(X)$.

A natural example of a local operator is given by the superposition operator

$$(h_f x)(\omega) = f(\omega, x(\omega)),$$

where $f : \Omega \times X \to Y$ is an $(\mathcal{F} \otimes Bor(X); Bor(Y))$-measurable function and $Bor(X)$ and $Bor(Y)$ are the σ-algebras of all Borel subsets of the spaces X and Y, respectively. Due to the above comment, the superposition operator can be regarded as a local operator on equivalence classes $h_f : \mathcal{P}(X) \to \mathcal{P}(Y)$.

It is well-known (see e. g. [4], [5]) that if $f : \Omega \times X \to Y$ is a Carathéodory function, i. e. $f(\cdot, x) \in \mathcal{P}(Y)$ for all $x \in X$ and $f(\omega, \cdot) : X \to Y$ is continuous for almost all $\omega \in \Omega$, then the superposition operator $h_f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is continuous in probability, i. e. with respect to the topologies of the spaces $\mathcal{P}(X)$ and $\mathcal{P}(Y)$.

However, not any local and continuous operator can be represented as a superposition operator generated by a Carathéodory function. The most famous example is the Itô integral [6], [7]. Thus, the class of superposition operators generated by Carathéodory functions is too poor for the theory of stochastic equations. On the other hand, stochastic integrals, superposition operators, their compositions and limits possess the property of locality [8]. Therefore, it was stated in the paper [3] that local and continuous in probability operators constitute a suitable class for developing a fixed point theory for stochastic analysis. It was, in particular, shown in [3] that there exists a stochastic version of Schauder's fixed point theorem, which is valid for certain local and continuous in probability operators. In the later publication [8] it was demonstrated that this fixed point theorem can be successfully applied to various stochastic differential and integral equations.

The proof of the stochastic counterpart of Schauder's theorem was based in [3] on a fixed point theorem for local operators in the spaces of finite dimensional random points $\mathcal{P}(R^n)$. This "stochastic Brouwer fixed point theorem" was justified in [3] for special subsets of these spaces, which was sufficient for many applications. However, some subsets naturally arising, for instance, in Malliavin calculus [9] were not covered, so that the problem of describing more general classes of invariant subsets of $\mathcal{P}(R^n)$, for which the stochastic Brouwer theorem is valid, was still highly relevant for applications, but remained open.

The following question is, therefore, discussed in the present paper: let Ξ be a closed, convex, bounded and nonempty subset of the set $\mathcal{P}(R^n)$ and $h : \Xi \to \Xi$ be a local and continuous in probability operator. For what subsets $\Xi \subset \mathcal{P}(R^n)$ the equation $hx = x$ has at least one solution? We give some examples of subsets, for which this stochastic Brouwer fixed point theorem holds true, but the central result of the paper states that it is, in general, false.

2. An example of a stochastic fixed point theorem

For some invariant subsets Ξ the answer to the above question is affirmative. To describe this class, let us consider a random subset $U : \omega \mapsto U(\omega)$ of R^n with the measurable graph $\operatorname{Gr} U \equiv \{(\omega, U(\omega)) \in \Omega \times R^n\} \in \mathcal{F} \otimes Bor(R^n)$. The set $\mathcal{P}(U)$ consists of all equivalence classes $[x]$ from $\mathcal{P}(R^n)$, for which there exists a representative $x' \in [x]$ such that $x'(\omega) \in U(\omega)$ for all $\omega \in \Omega$. If $U(\omega) \subset R^n$ is a.s. bounded, closed or convex, then $\mathcal{P}(U)$ is respectively bounded, closed or convex in the space $\mathcal{P}(R^n)$. Recall that bounded subsets $\mathcal{B}$ of the space $\mathcal{P}(R^n)$ can be described as follows: for any $\varepsilon > 0$ there is $r > 0$ such that $P\{\omega \in \Omega : \ |x(\omega)| > r\} < \varepsilon$ for all $x \in \mathcal{B}$.

Theorem 2.1. *Suppose that $U : \Omega \to R^n$ is a closed, convex, bounded and nonempty random subset of R^n such that*

$$Gr U \in \mathcal{F} \otimes Bor(R^n).$$

Let $\Xi = \mathcal{P}(U)$ and $h : \ \Xi \to \Xi$ be a continuous and local operator. Then h has at least one fixed point in Ξ.

Proof. By the main result of the paper [5], there exists a Carathéodory function $f : \mathrm{Gr}\, U \to R^n$ such that $h = h_f$. Evidently, $f(\omega, \cdot)$ leaves the set $U(\omega)$ a.s. invariant. By the deterministic Brouwer fixed point theorem, the set $Fix(\omega)$ consisting of all fixed points $x_\omega \in U(\omega)$ of the map $f(\omega, \cdot) : \ U(\omega) \to U(\omega)$ is a.s. nonempty. On the other hand, the function $F(\omega, x) = f(\omega, x) - x$ is Carathéodory and hence $\mathcal{F} \otimes Bor(R^n)$-measurable [10]. Therefore,

$$\{(\omega, Fix(\omega)) : \ \omega \in \Omega\} = G^{-1}(0) \in \mathcal{F} \otimes Bor(R^n)$$

and by the measurable selection theorem [10] there exists a $\mathcal{F}$-measurable function $x(\omega) \in U(\omega)$, i. e. a random point $x \in \mathcal{P}(U)$, such that $x(\omega) \in Fix(\omega)$ a.s. By construction, $hx = h_f x = x$ a.s., so that the equivalence class of x is a fixed point of the operator h. □

Remark 2.1. The most difficult part of the above proof is to justify the existence of a Carathéodory function f. This result is known as the generalized Nemytskii conjecture [5]. The conjecture itself says [11] that if a superposition operator $h_g : \mathcal{P}(R^n) \to \mathcal{P}(R^m)$ is continuous in probability, then g must satisfy the Carathéodory conditions. This conjecture is, unfortunately, not true in this formulation, but as it shown in [5], there always exists a Carathéodory function f such that $h_f = h_g$, and this result can be also extended to arbitrary local, continuous in probability operators and arbitrary separable metric spaces. The proof offered in [5] was based on projective approximations of metric spaces by topological T_0-spaces with finitely many points. An alternative proof for the simpler case of $Y = R^n$ and separable Banach spaces X can be found in the later publication [12]. This proof utilized special variational techniques.

Remark 2.2. Theorem 2.1 can be extended to some more general convex, closed and bounded subsets of the space $\mathcal{P}(R^n)$ that are relevant for stochastic analysis, see the paper [3] for details. However, the theorem in the next section shows that the stochastic Brouwer fixed point theorem for local operators is, in general, not valid for arbitrary closed, convex, bounded and nonempty subsets consisting of random points in finite dimensional spaces.

3. The counterexample

The proof of the main result of this section is based on a technical lemma.

Lemma 3.1. *Let*

$$(\Omega, \mathcal{F}, P) = (\Omega^1 \times \Omega^2, \mathcal{F}^1 \otimes \mathcal{F}^2, P^{(1)} \otimes P^{(2)})$$

be the product of two complete probability spaces and let $\Delta \in \mathcal{F}^1 \otimes \mathcal{F}^2$ have the property $P^{(2)}(\Delta(\omega^1))$ and $P^{(1)}(\Delta(\omega^2))$ is 0 or 1 for almost all $\omega^1 \in \Omega^1$ and $\omega^2 \in \Omega^2$, respectively.

Then $(P^{(1)} \otimes P^{(2)})(\Delta) = 0$ or 1.

Here $\Delta(\omega^1) = \{\omega^2 \in \Omega^2 : \ (\omega^1, \omega^2) \in \Delta\}$ and $\Delta(\omega^2) = \{\omega^1 \in \Omega^1 : \ (\omega^1, \omega^2) \in \Delta\}$.

P r o o f. Denoting $P = P^{(1)} \otimes P^{(2)}$ we define

$$\Delta_1 = \{\omega^1 \in \Omega^1 : \ P^{(2)}(\Delta(\omega^1)) > 0\} \in \mathcal{F}_1 \ \text{ and } \ \Delta_2 = \{\omega^2 \in \Omega^2 : \ P^{(1)}(\Delta(\omega^2)) > 0\} \in \mathcal{F}_2.$$

Then by the assumptions,

$$\Delta_1 = \{\omega^1 \in \Omega^1 : \ P^{(2)}(\Delta(\omega^1)) = 1\} \ \text{ and } \ \Delta_2 = \{\omega^2 \in \Omega^2 : \ P^{(1)}(\Delta(\omega^2)) = 1\}$$

and by Fubini's theorem $P\Delta = \int_{\Delta_1} P^{(2)}(\Delta(\omega^1))dP^{(1)} = P^{(1)}(\Delta_1)$, and similarly, $P\Delta = P^{(2)}(\Delta_2)$. On the other hand,

$$P(\Delta-(\Delta_1\times\Delta_2)) = P(\Delta-((\Delta_1\times\Omega_2)\cup(\Omega_1\times\Delta_2))) \le P(\Delta-(\Delta_1\times\Omega_2))+P(\Delta-(\Omega_1\times\Delta_2)) = 0,$$

so that $P(\Delta) \le P(\Delta_1 \times \Delta_2) = P(\Delta_1)P(\Delta_2) = (P(\Delta))^2$. Hence $P(\Delta) = 0$ or 1. □

Theorem 3.1. *There exists a closed, convex, bounded and nonempty subset Ξ of the space $\mathcal{P}(R^2)$ and a local and continuous in probability operator $h: \ \Xi \to \Xi$ such that the equation $hx = x$ has no solutions.*

P r o o f. The proof of the theorem consists of two parts. In the first part, we define the set Ξ and describe its properties, while the operator h will be constructed in the second part.

Part 1. Let $\mathbf{C}$ be the set of all complex numbers, $D = \{z \in \mathbf{C} : \ |z| < a\}$, where $a = \pi^{-0.5}$, so that the area of the circle is 1. Define $\Omega_k = \prod_{i=1}^{k} D_i$ and $\Omega^k = \prod_{i=k+1}^{\infty} D_i$, where $D_i = D$ ($i \ge 1$), $P_k = \bigotimes_{i=1}^{k} \mu_i$ and $P^k = \bigotimes_{i=k+1}^{\infty} \mu_i$, where μ_i is the Lebesgue measure on D_i. For brevity, we denote

$$\Omega = \Omega^0 = \prod_{i=1}^{\infty} D_i, \quad P = P^0 = \bigotimes_{i=1}^{\infty} \mu_i$$

and let $\mathcal{F}$ be the completion of the Borel σ-algebra on Ω with respect to P. This gives a complete probability space $(\Omega, \mathcal{F}, P)$.

We will construct Ξ as a subset of the space $\mathcal{P}(\mathbf{C})$, which can be identified with the space $\mathcal{P}(R^2)$.

Let E be the expectation, i. e. the integral with respect to the measure P. Consider the set $L^2 \subset \mathcal{P}(\mathbf{C})$ consisting of all square-integrable complex functions. The topology in L^2 is induced by the inner product $\langle x, y\rangle = Ex\bar{y}$. The set Ξ is defined to consist of all functions $x \in \mathcal{P}(\mathbf{C})$ that a.s. take their values in the closure $\bar{D}$ of the set D and satisfy the following property: for every $k \in N$ and every $z^k \in \Omega^k$ the function $x(\cdot, z^k)$ is holomorphic on Ω_k. We shall prove three following properties of the set Ξ:

1. Ξ is a closed, convex, bounded and nonempty subset of $\mathcal{P}(\mathbf{C})$;
2. Ξ is noncompact;
3. if $x, y \in \Xi$, then $P\{x = y\} = 0$ or 1.

Proof of Property (1). The set Ξ is by construction convex and bounded in $\mathcal{P}(\mathbf{C})$, the function $x(\omega) = x(z_1, z^1) = z_1$ ($z_1 \in \Omega_1$, $z^1 \in \Omega^1$) belongs to Ξ, so that $\Xi \ne \emptyset$. Let us prove that Ξ is closed in $\mathcal{P}(\mathbf{C})$. Pick a sequence $\{x_n\} \subset \Xi$, $x_n \to x$ in probability. Using an appropriate subsequence we may assume, without loss of generality, that $x_n(\omega) \to x(\omega)$ on a set A of full measure ($PA = 1$). Let $k \in N$ be an arbitrary number. Let $A(z^k) = \{z_k \in \Omega_k : \ (z_k, z^k) \in A\}$. By Fubini's theorem, the set $\hat{\Omega}^k$, which consists of all $z^k \in \Omega^k$ such that $P_k A(z^k) = 1$, has measure 1. Taking an arbitrary $z^k \in \hat{\Omega}^k$, let us consider the k-dimensional torus $\Gamma_\rho = \prod_{i=1}^k \gamma_{\rho_i}$, where $\rho = (\rho_1, ..., \rho_k)$, $\rho_i < a$ and $\gamma_{\rho_i} = \{z \in \mathbf{C} : \ |z| = \rho_i\}$. Let ν be the Lebesgue measure on Γ_ρ. Using again Fubini's theorem yields a set of $\rho \in [0, a) \times ... \times [0, a)$ of full Lebesgue measure, where $\nu\Gamma_\rho = \nu(\Gamma_\rho \cap A(z^k))$. In particular, there exists a sequence $\rho^m \to (a, ..., a)$ such that

$$\nu(\Gamma_{\rho^m}) = \nu(\Gamma_{\rho^m} \cap A(z^k)). \tag{3.1}$$

By construction, $x_n(\cdot, z^k) \to x(\cdot, z^k)$ ν-almost everywhere on each Γ_{ρ^m}.

Consider the integral

$$(2\pi i)^{-k} \int_{\Gamma_{\rho^m}} x(\xi_1, .., , \xi_k, z^k) \prod_{i=1}^k (\xi_i - \eta_i) d\xi \equiv \varphi_m(\eta_1, ..., \eta_k). \tag{3.2}$$

The integral exists for any $z_k = (\eta_1, ..., \eta_k)$ where $|\eta_i| < \rho_i^m$ ($i = 1, ..., k$, $m \in N$) and $\rho^m = (\rho_1^m, ..., \rho_k^m)$, as the integrand is bounded and measurable. By Hartogs' theorem [13], the functions $\varphi_m(\eta_1, ..., \eta_k)$ ($m \in N$) are holomorphic, i. e. complex differentiable, at these points, because they are holomorphic in each variable z_i:

$$\begin{aligned} &\tfrac{1}{\delta\eta_i} \left(\varphi_m(\eta_1, ..., \eta_i + \delta\eta_i, ..., \eta_k) - \varphi_m(\eta_1, ..., \eta_i, ..., \eta_k)\right) \\ &= \int_{\Gamma_{\rho^m}} y(\xi_1, ..., \xi_k, \eta_1, ..., \eta_k) \left((\xi_i - \eta_i - \delta\eta_i)^{-1} - (\xi_i - \eta_i)^{-1}\right) d\xi \\ &= \int_{\gamma_{\rho_i^m}} \left((\xi_i - \eta_i - \delta\eta_i)^{-1} - (\xi_i - \eta_i)^{-1}\right) d\xi_i \int_{\prod_{j \neq i} \gamma_{\rho_j^m}} y(\xi_1, ..., \xi_k, \eta_1, ..., \eta_k) d\xi_1 ... d\xi_{i-1} d\xi_{i+1} ... d\xi_k \ , \\ &\le C \max_{s \in \gamma_{\rho_i^m}} \tfrac{|\delta\eta_i|}{|s - \eta_i - \delta\eta_i||s - \eta_i|} = o(|\delta\eta_i|), \end{aligned}$$

for small $|\delta\eta_i|$ satisfying $|\eta_i + \delta\eta_i| \le b < \rho_i^m$. Here

$$y(\xi_1, ..., \xi_k, \eta_1, ..., \eta_k) = x(\xi_1, .., , \xi_k, z^k) \prod_{j \neq i}^k (\xi_j - \eta_j)$$

is a bounded function on Γ_{ρ^m}.

On the other hand, for every $z^k \in \hat{\Omega}^k$ the functions $x_n(\cdot, z^k)$ are holomorphic, so that applying the multivariate Cauchy formula and the Lebesgue convergence theorem yield

$$\begin{aligned} x_n(z_k, z^k) &= (2\pi i)^{-k} \textstyle\int_{\Gamma_{\rho^m}} x_n(\xi_1, .., , \xi_k, z^k) \prod_{i=1}^k (\xi_i - \eta_i) d\xi \equiv \varphi_m(\eta_1, ..., \eta_k) \\ &\to (2\pi i)^{-k} \textstyle\int_{\Gamma_{\rho^m}} x(\xi_1, .., , \xi_k, z^k) \prod_{i=1}^k (\xi_i - \eta_i) d\xi \equiv \varphi_m(\eta_1, ..., \eta_k) = \varphi_m(z_k) \ \text{ as } \ n \to \infty \end{aligned}$$

for each $m \in N$ and each $z_k = (\eta_1, ..., \eta_k)$ ($|\eta_i| < \rho_i^m$ ($i = 1, ..., k$). Therefore, $x(z_k, z^k) = \varphi_m(z_k)$ for almost all $z_k \in W_k^m \equiv \{(\eta_1, ..., \eta_k) : \ |\eta_i| < \rho_i^m, i = 1, ..., k$ and all $z^k \in \hat{\Omega}^k$, so that $x(\cdot, z^k)$ is holomorphic on any open set W_k^m and hence on the set Ω_k, because by construction, $\bigcap_{m=1}^\infty W_k^m = \Omega_k$. As $k \in N$ is arbitrary, we have proven that $x \in \Xi$, so that Ξ is closed in $\mathcal{P}(\mathbf{C})$.

Proof of Property (2). Consider the functions $x_k(\omega) = x(z_k, z^k) = x(\eta_1, ..., \eta_k, z^k) = \eta_k$ (here $z_k = (\eta_1, ..., \eta_k)$). Clearly, $x_k \in \Xi$. On the other hand, $\langle x_k, x_l \rangle = E\eta_k\bar{\eta}_l = \int_{D_k} \eta_k d\mu_k \int_{D_k} \bar{\eta}_l d\mu_l = 0$ if $k \neq l$, because

$$\int_{D_k} \eta_k d\mu_k = \int_D (u + iv) du dv = \int_0^{2\pi} d\theta \int_0^a r^2 (\cos\theta + i \sin\theta) dr = 0.$$

On the other hand, for all $k \in N$

$$\langle x_k, x_k\rangle = E\eta_k\bar{\eta}_k = \int_D (u^2+v^2)dudv = \int_0^{2\pi} d\theta \int_0^a r^3 dr = (2\pi)^{-1},$$

so that $\|x_k - x_l\|_{L^2} = \pi^{-1}$, $k \neq l$ and the sequence $\{x_k\}$ is not compact in the L^2-topology of the set Ξ. But $|x| \le a$ a.s. for all $x \in \Xi$. Therefore, the L^2-topology and the topology of $\mathcal{P}(\mathbf{C})$ are equivalent on Ξ, so that Ξ is not compact in the latter topology as well.

Proof of Property (3). It is sufficient to check that for any $x \in \Xi$, the measure of the set $\Gamma = \{\omega \in \Omega : \ x(\omega) = 0\}$ is either 0 or 1. Assume, on the contrary, that $0 < P\Gamma < 1$. By definition of the measure P as the product of linear Lebesgue measures, there always exist $k \in N$ and a Borel subset $B \subset \Omega_k$ such that $\Gamma \subset B \subset \Omega_k$ up to a 0-measure set and $P(B \times \Omega_k - \Gamma) < \frac{1}{2}P\Gamma$. Let $y = xI_{\Omega - B} \in \Xi$. By construction, $\{\omega \in \Omega : \ y(\omega) = 0\} = B \subset \Omega_k$ up to a 0-measure set and $0 < P(B) < 1$. Without loss of generality we may assume that y is holomorphic in z_k. In particular, y is holomorphic in each η_i on the set $D_i = D$, where $z_k = (\eta_1, ..., \eta_k)$. Therefore the Lebesgue measure of the set $\{\eta_i : \ z_k \in B\}$ is either 0 or 1 for any $(\eta_1, ..., \eta_{i-1}, \eta_{i+1}, ..., \eta_k) \in \prod_{j\neq i} D_i$.

Now, Property (3) follows from the induction argument and Lemma 3.1.

To conclude the first part of the proof, let us notice that any map defined on Ξ will be local due to property (3). Hence any continuous map $h : \ \Xi \to \Xi$ without a fixed point would satisfy all the assumptions of Theorem 3.1. This map will be constructed in

Part 2. Let us consider the polygonal chain P connecting the consecutive points $x_n \in \Xi$, which where defined in the course of the proof of Property (2). The set $\mathcal{C}$ is the union of the line segments $I_n \equiv [x_n, x_{n+1}] = \{y \in \Xi : \ y = \alpha x_n + (1-\alpha)x_{n+1}, \ 0 \le \alpha \le 1\}$. Assume that a sequence $\{y_\nu\} \subset \mathcal{C}$ converges to some $y \in \Xi$. As it has been mentioned, this is, in fact, the L^2-convergence, i. e. $E|y_\nu - y|^2 \to 0$ if $\nu \to \infty$. We claim that there exists $n \in N$ such that $y_\nu \in I_n$ for sufficiently large ν. To prove it we notice that if $m > n$ and $u = \alpha x_n + (1-\alpha)x_{n+1} \in I_n$ and $v = \beta x_m + (1-\beta)x_{m+1} \in I_m$ for some $\alpha, \beta \in [0,1]$, then

$$E|u-v|^2 = (2\pi)^{-1}\left(\alpha^2 + \beta^2 + (1-\alpha)^2 + (1-\beta)^2\right) \ge (2\pi)^{-1} \quad \text{if} \quad m - n \ge 2,$$
$$E|u-v|^2 = (2\pi)^{-1}\left(\alpha^2 + \beta^2 + (1-\alpha-\beta)^2 \ge \pi^{-1}\right) \quad \text{if} \quad m - n = 1,$$

due to orthogonality of x_n and the equality $E|x_n|^2 = (2\pi)^{-1}$. Therefore, if for some $\nu_0 \in N$ we have $E|y_\nu - y_{\nu_0}|^2 < \pi^{-1}$ for all $\nu \ge \nu_0$ and $y_{\nu_0} \in I_n$, then $y_\nu \in I_n$ as well for all $\nu \ge \nu_0$.

As each I_n compact, it implies that $y \in \mathcal{C}$, so that $\mathcal{C}$ is a closed subset of Ξ. On the other hand, the map $\eta : \mathcal{C} \to [0, \infty)$ defined on each I_n by $\eta(\alpha x_n + (1-\alpha)x_{n+1}) = n - \alpha$ is a bijection, because $I_n \cap I_m = \emptyset$ if $|m-n| \ge 2$ and $I_n \cap I_{n+1} = x_n$ for all $n \in N$. Let $y_\nu \to y$ in $\mathcal{C}$. Then there exists n such that $y_\nu = \alpha_\nu x_n + (1-\alpha_\nu)x_{n+1}$ for sufficiently large ν and $y = \alpha x_n + (1-\alpha)x_{n+1}$, where $\alpha = \lim\limits_{\nu\to\infty} \alpha_\nu$. Therefore, $\eta(y_\nu) = n - \alpha_\nu$ converges to $\eta(y) = n - \alpha$. Conversly, if $\eta(y_\nu)$ converges to $\eta(y) \in I_n$, then $\eta(y_\nu) \in I_n$ for sufficiently large ν. Therefore, $\eta(y_\nu) = n - \alpha_\nu$ and $\eta(y) = n - \alpha$, so that $\alpha = \lim\limits_{\nu\to\infty} \alpha_\nu$ and hence $y_\nu = \alpha_\nu x_n + (1-\alpha_\nu)x_{n+1} \to y = \alpha x_n + (1-\alpha)x_{n+1}$ as $\nu \to \infty$. We have proven that $\eta : \ \mathcal{C} \to [0,\infty)$ is a bijective homeomorphism.

The continuous map $h_0 : \ x \mapsto x + 1$ on $[0, \infty)$ has no fixed points. Topologically, the set $[0, \infty)$ is an absolute retract [14]. As its homeomorphic image $\mathcal{C}$ is closed in the metric space Ξ, there exists a retraction $\tau : \ \Xi \to \mathcal{C}$, i. e. a continuous map, for which $\tau(x) = x$ for all $x \in \mathcal{C}$. Put $h = \eta^{-1} h_0 \eta \tau : \Xi \to \mathcal{C} \subset \Xi$. This map is continuous in the L^2-topology and hence in the topology of the space $\mathcal{P}(\mathbf{C})$. On the other hand, if $hx = x$, then $x \in \mathcal{C}$, so that $h_0(\eta(x)) = \eta(x)$, where $\eta(x) \in [0, \infty)$, which cannot be the case. Therefore, the continuous operator $h : \Xi \to \Xi$ has no fixed points. $\square$

4. Conclusion

We have constructed a closed, convex, bounded and nonempty subset Ξ of the space $\mathcal{P}(R^2)$ and a local and continuous in probability operator $h : \Xi \to \Xi$, which has no fixed points. This provides a counterexample to what we called "the stochastic Brouwer fixed point theorem". The result means that a careful description of invariant subsets is needed in this theorem.

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Received 01.04.2021
Reviewed 03.06.2021
Accepted for press 10.06.2021

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Поступила в редакцию 01.04.2021
Поступила после рецензирования 03.06.2021
Принята к публикации 10.06.2021